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A New Four-Parameter Weibull Model for Lifetime Data

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We introduce a new four-parameter distribution with constant, decreasing, increasing, bathtub and upside-down bathtub failure rate called the transmuted exponentiated generalized Weibull model. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Rényi and Shannon entropies, order statistics and probability weighted moments are derived. The estimation of the model parameters is performed by maximum likelihood. The flexibility of the new model is illustrated by means of three applications to real data.

Keywords: Entropy, Exponentiated Generalized Family; Generating Function; Maximum Likelihood; Order Statistic; Transmuted Family.

2000 Mathematics Subject Classification: 60E05; 62E10

1. Introduction

Generalizing distributions is an old practice and has ever been considered as precious as other practical problems in statistics. There has been an increased interest to develop new methods for generating new families of distributions because there is a persistent need for extending the classical forms of the well-known distributions to be more capable for modeling data in different applied areas. One main objective for generalizing models is to explain how the lifetime phenomenon arises in fields like physics, computer science, public health, medicine, engineering, biology, industry, communications, life-testing and many others. For example, the well-known distributions such as exponential, Weibull and gamma are very limited in their characteristics and are unable to show wide flexibility.

The Weibull distribution is a very popular model and has been extensively used over the past decades for modelling data in engineering, reliability and biological studies. The need for extended forms of the Weibull model arises in many applied areas. The books by Murthy et al. (2004) and Nadarajah and Kotz (2005) provide some of the classical extensions of the Weibull model and some applications. It is the most popular distribution in the literature for analyzing lifetime data. However, its major drawback is that its hazard rate cannot accommodate bathtub and unimodal shapes, which are quite common in reliability and biological studies. So, many generalizations of the Weibull distribution have been proposed and studied to cope with these failure rates.

Among these models, we refer to the additive Weibull (Xie and Lai, 1995), exponentiated Weibull (Mudholkar et al., 1995, 1996), extended Weibull (Xie et al., 2002), modified Weibull (Lai et al., 2003), beta modified Weibull (Silva et al., 2010), Kumaraswamy Weibull (Cordeiro et al., 2010), transmuted Weibull (Aryal and Tsokos, 2011), Kumaraswamy modified Weibull (Cordeiro et al., 2012), transmuted complementary Weibull geometric (Afify et al., 2014), Marshall-Olkin additive Weibull (Afify et al., 2016) and Kumaraswamy transmuted exponentiated additive Weibull (Nofal et al., 2016) distributions.

In particular, the Weibull (W) distribution has probability density function (pdf) and cumulative distribution function (cdf) given (for $x \geq 0$) by

$$g(x;\beta) = \beta x^{\beta-1} \exp\left(-x^{\beta}\right) \qquad \text{and} \qquad G(x;\beta) = 1 - \exp\left(-x^{\beta}\right), \tag{1.1}$$

respectively, where $\beta > 0$ is a shape parameter. In this paper, we define and study a new lifetime model called the *transmuted exponentiated generalized Weibull* (TExGW) distribution. Its main feature is that three additional shape parameters are inserted in (1.1) to provide greater flexibility for the generated distribution. Based on the *transmuted exponentiated generalized-G* (TExG-G) family of distributions, we construct a new four-parameter TExGW model and give a comprehensive description of some of its mathematical properties hoping that it will attract wider applications in engineering, survival and lifetime data, reliability and other areas of research. Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative functions of the baseline model with parameter vector ξ. Yousof et al. (2015) defined the cdf of their TExG-G class by

$$F(x;\lambda,a,b,\xi) = \left[1-\overline{G}(x;\xi)^{a}\right]^{b}\left\{1+\lambda-\lambda\left[1-\overline{G}(x;\xi)^{a}\right]^{b}\right\}. \tag{1.2}$$

The pdf of the TExG-G model is given by

$$f(x;\lambda,a,b,\xi) = abg(x;\xi)\,\overline{G}(x;\xi)^{a-1}\left[1-\overline{G}(x;\xi)^{a}\right]^{b-1} \times\left\{1+\lambda-2\lambda\left[1-\overline{G}(x;\xi)^{a}\right]^{b}\right\}. \tag{1.3}$$

We define the TExG-G distribution with three extra positive parameter λ, a and b by (1.2) and (1.3). A random variable X with pdf (1.3) is denoted by $X \sim$TExG-G(λ, a, b, ξ). If $\lambda = 0$, the TExG-G model reduces to the exponentiated generalized-G family studied by Cordeiro et al. (2013). If $a = b = 1$, it corresponds to the transmuted class (TC) defined by Shaw and Buckley (2007). If $\lambda = 0$ and $b = 1$, the TExG-G model reduces to the exponentiated-G (Ex-G) family pioneered by Gupta et al. (1998). If $a = 1$, it gives the transmuted generalized class (TG-G) defined by Alizadeh et al. (2015) and, finally, the TExG-G model reduces to the baseline distribution when $a = b = 1$ and $\lambda = 0$.

This paper is outlined as follows. In Section 2, we define the TExGW distribution and provide some plots of its pdf and hazard rate function (hrf). Section 3 provides some mathematical properties including ordinary and incomplete moments, probability weighted moments (PWMs), order statistics and residual and reversed residual life functions. Characterizations of the new distribution based on truncated moments are addressed in Section 4. Applications and parameters estimation are discussed in Section 5. Finally, some concluding remarks are discussed in Section 6.

2. The TExGW Distribution

By inserting the cdf in (1.1) into Equation (1.2), we can write the cdf of the TExGW model (for $x > 0$) as

$$F(x;\lambda,a,b,\beta) = \left[1-\exp\left(-ax^{\beta}\right)\right]^{b}\left\{1+\lambda-\lambda\left[1-\exp\left(-x^{\beta}\right)\right]^{b}\right\}. \tag{2.1}$$

The corresponding pdf of Equation (2.1) is given by

$$f(x;\lambda,a,b,\beta) = ab\beta x^{\beta-1}\exp\left(-ax^{\beta}\right)\left[1-\exp\left(-ax^{\beta}\right)\right]^{b-1} \times\left\{1+\lambda-2\lambda\left[1-\exp\left(-ax^{\beta}\right)\right]^{b}\right\}. \tag{2.2}$$

where β, a and b are positive parameters and $|\lambda| \leq 1$. Here, a is a scale parameter and β, λ and b are shape parameters. We denote a random variable X having pdf (2.2) by $X \sim$TExGW(λ, a, b, β). Table 1 gives special TExGW models.

Table 1. Sub-models of the TExGW model

No.	Distribution	β	λ	a	b
1	ExGW	β	0	a	b
2	TExW	β	λ	1	b
3	TGW	β	λ	a	1
4	GW	β	0	1	b
5	TW	β	λ	1	1
6	ExW	β	0	1	b
7	W	β	0	1	1

From (2.1) and (2.2), the hrf of X is given by

$$h(x;\lambda,a,b,\beta) = \frac{ab\beta x^{\beta-1}\exp\left(-ax^{\beta}\right)\left[1-\exp\left(-ax^{\beta}\right)\right]^{b-1}}{1-\left[1-\exp\left(-ax^{\beta}\right)\right]^{b}\left\{1+\lambda-\lambda\left[1-\exp\left(-ax^{\beta}\right)\right]^{b}\right\}} \times\left\{1+\lambda-2\lambda\left[1-\exp\left(-ax^{\beta}\right)\right]^{b}\right\}.$$

Some plots of the TExGW density for selected parameter values are displayed in Figure 1. Figure 2 provides plots of the hrf of the TExGW model for some parameter values.

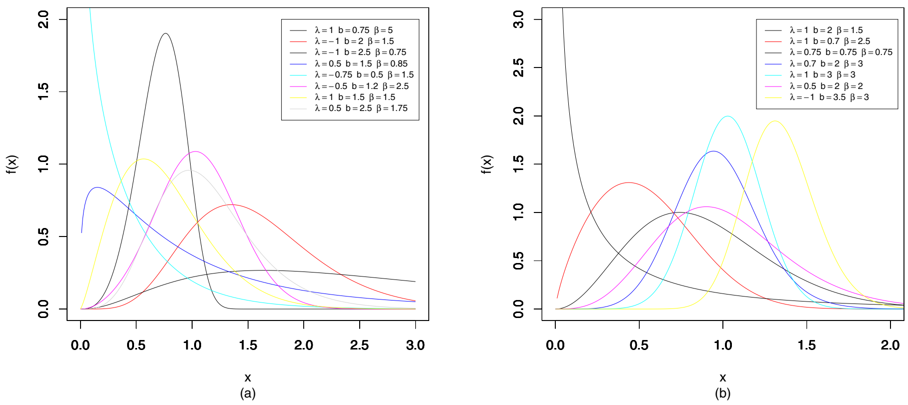

Fig. 1. Plots of the TExGW density function for some parameter values.

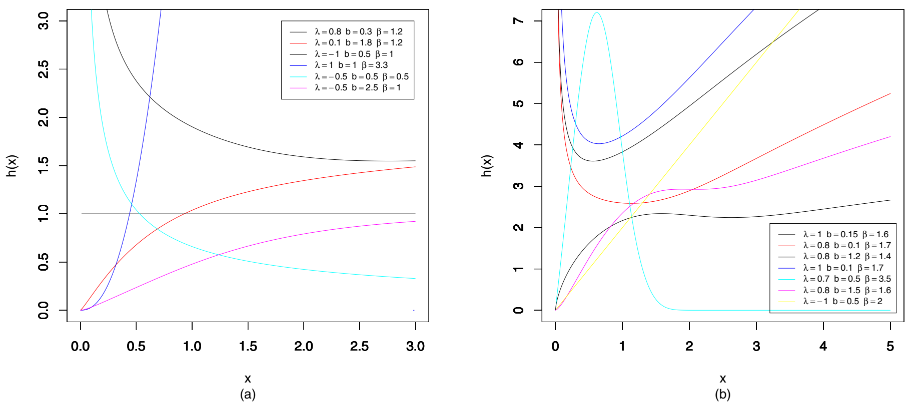

Fig. 2. Some plots of the TExGW hazard rate function

3. Properties

Let T be a random variable having the exponentiated Weibull (ExW) distribution with positive parameters β and a. Then, the pdf and cdf of T are given (for $t > 0$) by

$$g(t) = a\beta t^{\beta-1} \exp\left(-t^{\beta}\right) \left[1-\exp\left(-t^{\beta}\right)\right]^{a-1} \text{ and } G(t) = \left[1-\exp\left(-t^{\beta}\right)\right]^{a}.$$

The following lemma is useful to obtain several properties of the TExGW distribution.

Lemma 3.1. *The TExGW distribution can be expressed as an infinite series representation of the ExW distributions.*

Proof. Consider the power series

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\,\Gamma(b-j)} z^j, |z| < 1 \text{ and } b > 0 \text{ is a real non-integer.}$$

The cdf of X in (2.1) can be expressed as

$$F(x) = (1+\lambda) \sum_{j=0}^{\infty} (-1)^j \binom{b}{j} \left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]\right\}^{aj} - \lambda \sum_{j=0}^{\infty} (-1)^j \binom{2b}{j} \left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]\right\}^{aj}.$$

We can write

$$F(x) = \sum_{k,j=0}^{\infty} (-1)^{k+j} \binom{aj}{k} \left[(1+\lambda)\binom{b}{j} - \lambda\binom{2b}{j}\right] \left[1-\exp\left(-x^{\beta}\right)\right]^{k}.$$

Then,

$$F(x) = \sum_{k=0}^{\infty} t_k \,\Pi(x)^k, \tag{3.1}$$

where $\Pi(x)^k = [1-\exp\{-x^{\beta}\}]^k$ is the ExW cdf with power parameter k and

$$t_k = \sum_{j=0}^{\infty} (-1)^{k+j} \binom{aj}{k} \left[(1+\lambda)\binom{b}{j} - \lambda\binom{2b}{j}\right].$$

The corresponding TExGW density function is obtained by differentiating (3.1)

$$f(x) = \sum_{k=0}^{\infty} t_k \,\pi_k(x), \tag{3.2}$$

where $\pi_k(x)$ is the ExW pdf with power parameter k. □

Remark 3.1. For any $r > -\beta$, AL-Hussaini and Ahsanullah (2015) derived the rth ordinary and incomplete moments of T as

$$\mu_r' = \Gamma(1 + r/\beta) \sum_{j=0}^{\infty} \frac{c_j(a)}{(j+1)^{\frac{r}{\beta}+1}}$$

and

$$\varphi_r(t) = \gamma\left(1 + r/\beta, t^{-\beta}\right) \sum_{j=0}^{\infty} \frac{c_j(a)}{(j+1)^{\frac{r}{\beta}+1}},$$

respectively, where $c_j(a) = (-1)^j\, a\,(a-1)\ldots(a-j)\,/\,j!$, $\Gamma(a) = \int_0^\infty t^{a-1}\,\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ is the complete gamma function and $\gamma(a,x) = \int_0^x t^{a-1}\,\mathrm{e}^{-\mathrm{t}}\mathrm{dt}$ is the incomplete gamma function. For further information about the ExW distribution we refer to Mudholkar and Srivastava (1993) and Nadarajah and Kotz (2006).

3.1. *Moments*

The rth ordinary moment of X is given by

$$\mu_r' = E(X^r) = \int_0^\infty x^r\, f(x)\, dx.$$

Using (2.2) and Remark 1, we obtain

$$\mu_r' = \Gamma\left(\frac{r}{\beta} + 1\right) \sum_{k,j=0}^{\infty} \frac{c_j(k)\, t_k}{(j+1)^{\frac{r}{\beta}+1}}.$$

The nth central moment of X, say μ_n, can be expressed as

$$\mu_n = E(X - \mu_1')^n = \sum_{h=0}^{n} (-1)^h \binom{n}{h} (\mu_1')^n\, \mu_{n-h}'.$$

The cumulants (κ_n) of X can be determined recursively from $\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r\, \mu_{n-r}'$, where $\kappa_1 = \mu_1'$, etc. The skewness and kurtosis measures follow from the standardized third and fourth cumulants, respectively. The rth incomplete moment of X, say $\varphi_r(t)$, is given by

$$\varphi_r(t) = \int_0^t x^r\, f(x)\, dx.$$

Using Equation (3.2), we obtain

$$\varphi_r(t) = \gamma\left(\frac{r}{\beta} + 1, t^{-\beta}\right) \sum_{k,j=0}^{\infty} \frac{c_j(k)\, t_k}{(j+1)^{\frac{r}{\beta}+1}}.$$

In Table 2 we provide numerical values for the mean, variance, skewness and kurtosis of X for selected parameter values to illustrate their effects on these measures. The effects of the parameters on the mean, variance, skewness and kurtosis are displayed in Figures 3 and 4, respectively.

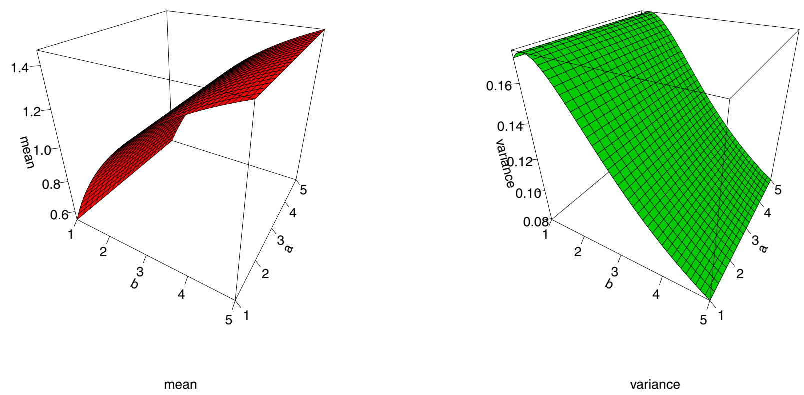

Fig. 3. Plots of the mean and variance for the TExGW distribution

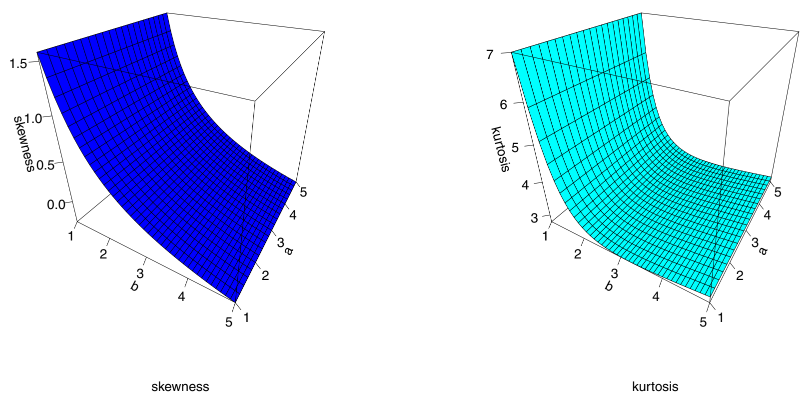

Fig. 4. Plots of the skewness and kurtosis for the TExGW distribution

3.2. *Probability weighted moments*

The (s,r)th PWM of X, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\left[X^s\, F(X)^r\right] = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\, dx.$$

Using Equations (2.1) and (2.2), we can write after some algebra

$$f(x)\; F(x)^r = \sum_{k=0}^{\infty} w_{k+1}\, \pi_{k+1}(x)\,,$$

Table 2. Mean, variance, skewness and kurtosis for selected parameter values

λ	b	β	mean	variance	skewness	kurtosis
-1	0.5	0.5	2	19.9999	6.6188	87.7200
		1.5	0.9027	0.3757	1.0720	4.3904
		5	0.9182	0.0442	-0.2541	2.8803
	1.5	0.5	4.7222	45.4969	4.4883	42.4603
		1.5	1.4361	0.3671	0.8035	3.9518
		5	1.0936	0.0203	-0.0830	3.0094
	5	0.5	10.1286	92.5370	3.2954	24.5719
		1.5	2.0092	0.3125	0.7685	4.0150
		5	1.2231	0.0102	0.1684	3.0676
-0.5	0.5	0.5	1.5434	15.7237	7.4461	109.8291
		1.5	0.7528	0.3741	1.1880	4.6235
		5	0.8420	0.0637	-0.3672	2.8183
	1.5	0.5	3.7581	37.4831	4.9058	49.9351
		1.5	1.2661	0.4036	0.7624	3.7966
		5	1.0425	0.0291	-0.3467	3.2360
	5	0.5	8.4029	80.7202	3.4641	26.8255
		1.5	1.8467	0.3552	0.6484	3.7814
		5	1.1888	0.0137	-0.0765	3.1440
0.5	0.5	0.5	0.6301	5.9201	11.8557	275.7817
		1.5	0.4531	0.2360	1.8864	7.6699
		5	0.6896	0.0677	0.0245	2.5260
	1.5	0.5	1.8298	15.8779	7.2179	105.9518
		1.5	0.9261	0.3032	1.1583	4.9776
		5	0.9403	0.0312	-0.0966	3.0123
	5	0.5	4.9515	39.2184	4.6769	47.2232
		1.5	1.5217	0.2822	0.9306	4.5658
		5	1.1202	0.0136	0.1351	3.1804
1	0.5	0.5	0.1735	0.3928	11.1446	247.7442
		1.5	0.3032	0.0996	1.7516	6.9258
		5	0.6134	0.0524	-0.0262	2.4854
	1.5	0.5	0.8656	2.2866	5.0914	52.7148
		1.5	0.7560	0.1662	0.8334	3.8427
		5	0.8892	0.0245	-0.2767	3.0333
	5	0.5	3.2258	9.5333	2.8168	18.1678
		1.5	1.3592	0.1665	0.5268	3.4537
		5	1.0859	0.0100	-0.1238	3.0852

where

$$w_{k+1} = \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k} ab\lambda^i (1+\lambda)^{r-i}}{(k+1)} \binom{r}{i} \binom{a(j+1)-1}{k} \times \left[(1+\lambda) \binom{b(r+2)-1}{j} - 2\lambda \binom{b(r+3)-1}{j} \right].$$

Then, the (s,r)th PWM of X can be expressed as

$$\rho_{s,r}=\Gamma\left(\frac{s}{\beta}+1\right)\sum_{k,\ell=0}^{\infty}\frac{c_l\left(k+1\right)w_{k+1}}{\left(\ell+1\right)^{\frac{s}{\beta}+1}}.$$

3.3. *Order statistics*

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1,\ldots,X_n$ be a random sample from the TExGW model of distributions and let $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. The pdf of the ith order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}\left(x\right)=\frac{f\left(x\right)}{\mathrm{B}\left(i,n-i+1\right)}\sum_{j=0}^{n-i}\left(-1\right)^{j}\binom{n-i}{j}F^{j+i-1}\left(x\right), \tag{3.3}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (2.1) and (2.2) in Equation (3.3), we obtain

$$f\left(x\right)F\left(x\right)^{j+i-1}=\sum_{k=0}^{\infty}d_{k+1}\;\pi_{k+1}\left(x\right),$$

where

$$d_{k+1}=\frac{ab\left(-1\right)^{k}}{k+1}\sum_{m,w=0}^{\infty}\frac{\left(-1\right)^{m+w}\lambda^{m}}{\left(1+\lambda\right)^{-\left(j+i-m-1\right)}}\binom{j+i-1}{m}\binom{a\left(w+1\right)-1}{k}$$
$$\times\left[\left(1+\lambda\right)\binom{b\left(j+i+1\right)-1}{w}-2\lambda\binom{b\left(j+i+2\right)-1}{w}\right].$$

Let Y_{k+1} be a random variable with the ExW pdf $\pi_{k+1}(x)$. Further, the pdf of $X_{i:n}$ can be rewritten as

$$f_{i:n}\left(x\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{\left(-1\right)^{j}}{\mathrm{B}\left(i,n-i+1\right)}\binom{n-i}{j}d_{k+1}\,\pi_{k+1}\left(x\right).$$

Then, the density function of the TExGW order statistics is a mixture of ExW densities. This equation reveals that the properties of $X_{i:n}$ follow from those properties of Y_{k+1}. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^{q}\right)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{\left(-1\right)^{j}}{\mathrm{B}\left(i,n-i+1\right)}\binom{n-i}{j}d_{k+1}\,E\left(Y_{k+1}^{q}\right).$$

Then, the qth moment of $X_{i:n}$ is given by

$$E\left(X_{i:n}^{q}\right)=\Gamma\left(\frac{q}{\beta}+1\right)\sum_{k,\ell=0}^{\infty}\sum_{j=0}^{n-i}\frac{\left(-1\right)^{\ell+j}\left(k+1\right)d_{k+1}}{\mathrm{B}\left(i,n-i+1\right)\left(\ell+1\right)^{\frac{q}{\beta}+1}}\binom{k}{\ell}\binom{n-i}{j}. \tag{3.4}$$

Based upon the moments in Equation (3.4), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable TExGW order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r=\frac{1}{r}\sum_{d=0}^{r-1}\left(-1\right)^{d}\binom{r-1}{d}E\left(X_{r-d:r}\right),\; r\geq 1.$$

3.4. *Residual and Reversed Residual Lifes*

The nth moment of the residual life, say $m_n(t) = E[(X-t)^n \mid X > t]$, $n = 1, 2, \ldots$, uniquely determine $F(x)$. The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{R(t)} \int_t^\infty (x-t)^n dF(x).$$

We can write

$$\begin{aligned} m_n(t) &= \frac{1}{R(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} (-t)^{n-r} t_k \binom{n}{r} \int_t^\infty x^r \, \pi_k(x)\, dx \\ &= \frac{1}{R(t)} \sum_{k,j=0}^{\infty} \sum_{r=0}^{n} \frac{k\, c_j(k) t_k (-t)^{n-r}}{(j+1)^{\frac{r}{\beta}+1}} \binom{n}{r} \gamma\left(\frac{r}{\beta}+1, t^{-\beta}\right). \end{aligned}$$

The mean residual life (MRL), or the life expectation at age t, of X can be obtained by setting $n = 1$ in the last equation and it represents the expected additional life length for a unit which is alive at age t. The nth moment of the reversed residual life, say $M_n(t) = E\left[(t-X)^n \mid X \le t\right]$, for $t > 0$ and $n = 1, 2, \ldots$, uniquely determines $F(x)$. We have

$$M_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n dF(x).$$

Then, the nth moment of the reversed residual life of X becomes

$$\begin{aligned} M_n(t) &= \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t_k t^{n-r} \int_0^t x^r \, \pi_k(x)\, dx \\ &= \frac{1}{F(t)} \sum_{k,j=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r k\, c_j(k) t_k t^{n-r}}{(j+1)^{\frac{r}{\beta}+1}} \binom{n}{r} \gamma\left(\frac{r}{\beta}+1, t^{-\beta}\right). \end{aligned}$$

The mean inactivity time (MIT) also called the mean reversed residual life function represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$ and it is defined by $M_1(t) = E[(t-X) \mid X \le t]$. The MIT of the TExGW model can be obtained easily by setting $n = 1$ in the above equation.

4. Characterization results

Consider the following lemmas from Ahsanullah and Alzaatreh (2016).

Lemma 4.1. *Let X be an absolutely continuous random variable having cdf $F(x)$ and pdf $f(x)$ with $0 \le x < \infty$. Assume that $E(X)$ is finite and $E(X|X \le x) = g(x)\,\tau(x)$, where $g(x)$ is a continuous differentiable function on $[0,\infty)$ and $\tau(x) = f(x)/F(x)$. Then,*

$$f(x) = c \exp\left(\int_0^x \frac{u - g'(u)}{g(u)} du\right),$$

where c is determined by the condition $\int_0^\infty f(x)\,dx = 1$.

Lemma 4.2. *Let X be an absolutely continuous random variable having cdf $F(x)$ and pdf $f(x)$ with $0 \le x < \infty$. Assume that $E(X)$ is finite and $E(X|X \ge x) = g(x)\,h(x)$, where $g(x)$ is a continuous* *differentiable function on* $[0,\infty)$ *and* $h(x) = f(x)/[1-F(x)]$. *Then,*

$$f(x) = c\exp\left(\int_0^x \frac{u+g'(u)}{g(u)}du\right),$$

where c is determined by the condition $\int_0^\infty f(x)\,dx = 1$.

Theorem 4.1. *Let X be an absolutely continuous random varriable with cdf* $F(x)$ *and pdf* $f(x)$. *We assume that* $0 = inf\{X|F(x)>0\}$, $\infty = Sup\{X|F(x)<1\}$, $E(X)$ *exists and* $f(x)$ *is differentiable for all x,* $0<x<\infty$. *Then* $E(X|X\le x) = g(x)\tau(x)$, *where*

$$g(x) = \frac{x\left[1-\exp\left(-x^{\beta}\right)\right]^{k} + \sum_{j=0}^{\infty}\frac{(-1)^{j}c_j}{\beta j^{\frac{1}{\beta}}}\Gamma\left(jx^{\beta},\frac{1}{\beta}\right)}{k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1}}, \tau(x) = \frac{f(x)}{F(x)},$$

$$c_j = \begin{cases} \frac{k!}{j!(k-j)!} & \text{if } k \text{ is an integer,} \\ \frac{k(k-1)\ldots(k-j-1)}{j!} & \text{if } k \text{ is not an integer,} \\ 1 & \text{if } j=0, \end{cases}$$

and $\Gamma(p,n) = \int_0^p x^{n-1}\exp(-x)\,dx$ *if and only if X has the pdf* $f(x) = k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1}$.

Proof. Setting

$$f(x) = k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1},$$

we have

$$\begin{aligned} f(x)g(x) &= \int_0^x k\beta u^{\beta}\exp\left(-u^{\beta}\right)\left[1-\exp\left(-u^{\beta}\right)\right]^{k-1}du \\ &= x\left[1-\exp\left(-x^{\beta}\right)\right]^{k} - \sum_{j=0}^{\upsilon} c_j(-1)^j\int_0^x \exp\left(-ju^{\beta}\right)du, \end{aligned}$$

where c_j is defined above.
and

$$\nu = \begin{cases} k & \text{if } k \text{ is an integer,} \\ \infty & \text{if } k \text{ is not an integer.} \end{cases}$$

Since

$$\int_0^x \exp\left\{-ju^{\beta}\right\}du = \int_0^{jx^{\beta}}\frac{1}{\beta j^{\frac{1}{\beta}}}t^{\frac{1}{\beta}-1}\exp(-t)\,dt = \frac{1}{\beta j^{\frac{1}{\beta}}}\Gamma\left(jx^{\beta},\frac{1}{\beta}\right).$$

Thus

$$g(x) = \frac{x\left[1-\exp\left(-x^{\beta}\right)\right]^{k} - \sum_{j=0}^{\upsilon} \frac{c_j(-1)^j}{\beta j^{\frac{1}{\beta}}}\Gamma\left(jx^{\beta}, \frac{1}{\beta}\right)}{k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1}}.$$

Now suppose that

$$g(x) = \frac{x\left[1-\exp\left(-x^{\beta}\right)\right]^{k} - \sum_{j=0}^{\upsilon} \frac{c_j(-1)^j}{\beta j^{\frac{1}{\beta}}}\Gamma\left(jx^{\beta}, \frac{1}{\beta}\right)}{k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1}}.$$

Then one can show easily

$$\frac{x-g'(x)}{g(x)} = \frac{\beta-1}{x} - \beta x^{\beta-1} + \frac{(k-1)\beta x^{\beta-1}\exp\left(-x^{\beta}\right)}{1-\exp\left(-x^{\beta}\right)}.$$

By using Lemma 1

$$\frac{f'(x)}{f(x)} = \frac{\beta-1}{x} - \beta x^{\beta-1} + \frac{(k-1)\beta x^{\beta-1}\exp\left\{-x^{\beta}\right\}}{1-\exp\left\{-x^{\beta}\right\}}.$$

Integrating the above equation, we have

$$f(x) = cx^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1},$$

where c is a constant. Now, using the condition $\int_0^{\infty} f(x)\,dx = 1$, we obtain

$$f(x) = k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1}.$$

□

Theorem 4.2. *Let X be an absolutely continuous random varriable with cdf* $F(x)$ *and pdf* $f(x)$*. We assume that* $0 = inf\{X|F(x) > 0\}$, $\infty = Sup\{X|F(X) < 1\}$, $E(X)$ *exists and* $f(x)$ *is differentiable for all x,* $0 < x < \infty$*. Then* $E(X|X \geq x) = h(x)\,r(x)$*, where*

$$h(x) = \frac{\sum_{j=0}^{\upsilon} \frac{kc_j(-1)^j}{(1+J)^{1/}}\Gamma\left(1+\frac{1}{\beta}\right) - x\left[1-\exp\left(-x^{\beta}\right)\right]^{k} - \sum_{j=0}^{\infty} \frac{(-1)^j c_j}{\beta j^{\frac{1}{\beta}}}\Gamma\left(jx^{\beta}, \frac{1}{\beta}\right)}{k\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{k-1}},$$

$$r(x) = \frac{f(x)}{1-F(x)},$$

$$c_j = \begin{cases} \frac{k!}{j!(k-j)!} & \text{if } k \text{ is an integer}, \\ \frac{k(k-1)\ldots(k-j+1)j}{j!} & \text{if } k \text{ is not an integer}, \\ 1 & \text{if } j=0, \end{cases}$$

$\Gamma(n) = \int_0^\infty x^{n-1}\exp(-x)\,dx$ *and* $\Gamma(p,n) = \int_0^p x^{n-1}\exp(-x)\,dx$. *If and only if X has pdf* $f(x) = k\beta x^{\beta-1}\exp\left(-x^\beta\right)\left[1-\exp\left(-x^\beta\right)\right]^{k-1}$.

Proof. Similar to the proof of Theorem 4.1. □

5. Estimation and Applications

The maximum likelihood is the most commonly estimation method employed in Statistics. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals for the model parameters. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. In this section, we use the maximum likelihood to estimate the model parameters and compare the fits of the TExGW model and other lifetime distributions.

5.1. *Parameter Estimation*

Let $x_1,\ldots,x_n$ be a random sample from the TExGW distribution with parameters λ, a, b and β. Let $\theta = (\lambda, a, b, \beta)^T$ be the parameter vector. For determining the MLE of θ, the log-likelihood function $\ell = \ell(\theta)$ is given by

$$\ell = n(\log a + \log b + \log\beta) + (\beta-1)\sum_{i=1}^n \log(x_i) - a\sum_{i=1}^n x_i^\beta + (b-1)\sum_{i=1}^n \log s_i + \sum_{i=1}^n \log\left(1+\lambda-2\lambda s_i^b\right),$$

where $s_i = (1-t_i)$ and $t_i = \exp(-a x_i^\beta)$.

The log-likelihood $\ell(\theta)$ for θ can be maximized either directly by using the `R` (`optim` function), `SAS(PROC NLMIXED)`, `Ox` program (`MaxBFGS` sub-routine) or by solving the nonlinear likelihood equations obtained by differentiating the log-likelihood.

The components of the score vector, $\mathbf{U}(\theta) = \frac{\partial\ell}{\partial\theta} = \left(\frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\beta}\right)^T$, are given by

$$U_\lambda = \sum_{i=1}^n \frac{1-2s_i^b}{\left(1+\lambda-2\lambda s_i^b\right)}, U_b = \frac{n}{b} + \sum_{i=1}^n \log s_i - 2\lambda\sum_{i=1}^n \frac{s_i^b \log s_i}{\left(1+\lambda-2\lambda s_i^b\right)},$$

$$U_a = \frac{n}{a} - \sum_{i=1}^n x_i^\beta + (b-1)\sum_{i=1}^n \frac{x_i^\beta}{s_i t_i} - 2\lambda\sum_{i=1}^n \frac{b s_i^{b-1} x_i^\beta}{\left(1+\lambda-2\lambda s_i^b\right) t_i}$$

and

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) + a(b-1) \sum_{i=1}^{n} \frac{x_i^{\beta} \log(x_i)}{s_i t_i} - a \sum_{i=1}^{n} \frac{\log(x_i)}{x_i^{-\beta}}$$
$$- 2\lambda ab \sum_{i=1}^{n} \frac{s_i^{b-1} x_i^{-\beta} \log(x_i)}{\left(1+\lambda-2\lambda s_i^{b}\right) t_i}.$$

Setting the nonlinear system of equations $U_\lambda = U_a = U_b = U_\beta = 0$ and solving them simultaneously yields the MLE $\widehat{\theta} = (\widehat{\lambda}, \widehat{a}, \widehat{b}, \widehat{\beta})^T$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 4×4 observed information matrix $J(\theta) = \{-\frac{\partial^2 \ell}{\partial r \partial s}\}$ (for $r, s = \lambda, a, b, \beta$), whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\theta}$ can be approximated by a multivariate normal $N_4(0, J(\widehat{\theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\theta})$ is the total observed information matrix evaluated at $\widehat{\theta}$.

5.2. *Real Data Analysis*

In this section, we provide three applications to real data sets to illustrate the importance and potentiality of the TExGW distribution.

Data Set I: Waiting Times in a Bank

The first data set (Ghitany et al., 2008) consists of 100 observations on waiting time (in minutes) before the customer received service in a bank.

Data Set II: Cancer Patients

The second data set corresponds to the remission times (in months) of 128 bladder cancer patients (Lee and Wang, 2003). These data were used by Mead and Afify (2016) to fit the Kumaraswamy exponentiated Burr XII distribution.

Data Set III: Exceedances of Wheaton River Flood

The third data set represents the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place.

For the three data sets, we shall compare the fits of the TExGW distribution with other competitive distributions, namely: the modified beta Weibull (MBW) (Khan, 2015), transmuted Weibull Lomax (TWL) (Afify et al., 2015) and generalized transmuted Weibull (GTW) (Nofal et al., 2017) distributions. The pdfs of these distributions are given by (for $x > 0$):

McW: $f(x) = \frac{\beta c \alpha^\beta}{B(a/c,b)}\, x^{\beta-1}\, e^{-(\alpha x)^\beta} \left[1 - e^{-(\alpha x)^\beta}\right]^{a-1} \times \left\{1 - \left[1 - e^{-(\alpha x)^\beta}\right]^{c}\right\}^{b-1}$;

MBW: $f(x) = \frac{\beta \alpha^{-\beta} c^{a}}{B(a/c,b)}\, x^{\beta-1}\, e^{-b\left(\frac{x}{\alpha}\right)^\beta} \left[1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}\right]^{a-1} \times \left\{1 - (1-c)\left[1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}\right]^{c}\right\}^{-a-b}$;

TWL: $f(x) = \frac{ab\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{b\alpha-1} e^{-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}} \left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]^{b-1} \times \left\{1 - \lambda + 2\lambda e^{-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}}\right\}$;

GTW: $f(x) = \beta \alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^\beta} \left[1 - e^{-(\alpha x)^\beta}\right]^{a-1} \times \left\{a(1+\lambda) - \lambda(a+b)\left[1 - e^{-(\alpha x)^\beta}\right]^{b}\right\}$.

The parameters of the above pdfs are all positive real numbers except for the TWL and GTW distributions for which $|\lambda| \leq 1$.

In order to compare the fitted models, we consider the following goodness-of-fit statistics: the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$), consistent Akaike information criterion ($CAIC$), $-2\widehat{\ell}$, Anderson-Darling (A^*) and Cramér-von Mises (W^*), where $\widehat{\ell}$ denotes the maximized log-likelihood. Generally, the smaller these statistics are, the better the fit.

Table 3 lists the values of $-2\widehat{\ell}$, AIC, BIC, $HQIC$, $CAIC$, W^* and A^*, whereas the MLEs of the model parameters and their corresponding standard errors are given in Table 4. These numerical results are obtained using `MATHCAD PROGRAM`.

Table 3. Goodness-of-fit statistics for the data sets I, II and III.

Model	$-2\widehat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
Data set I							
TExGW	634.43	642.43	642.851	646.647	652.851	0.01814	0.12799
TWL	634.725	644.725	645.364	649.997	657.751	0.01936	0.13445
MBW	637.248	647.248	647.886	652.519	660.273	0.04528	0.34588
GTW	641.837	651.837	652.475	657.109	664.863	0.0916	0.66481
Data set II							
TExGW	819.846	827.846	828.171	832.481	839.254	0.03212	0.20757
TWL	820.402	830.402	830.894	836.196	844.662	0.03377	0.22162
GTW	821.347	831.347	831.839	837.141	845.607	0.04691	0.30583
MBW	828.027	838.027	838.519	843.821	852.288	0.10679	0.72074
Data set III							
TExGW	507.053	515.053	515.65	518.678	524.16	0.19123	1.06515
TWL	512.787	522.787	523.696	527.318	534.17	0.22213	1.3025
MBW	512.008	522.008	522.917	526.54	533.391	0.24923	1.41098
GTW	513.321	523.321	524.23	527.853	534.704	0.28282	1.57651

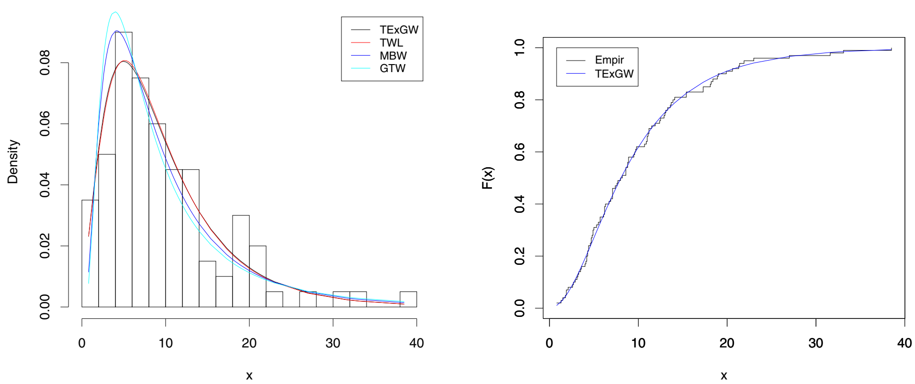

Fig. 5. The fitted pdfs (left panel) and the TExGW fitted cdf (right panel) for data set I

Table 3 compares the fits of the TExGW distribution with the TWL, MBW and GTW distributions. The results in this table reveal that the TExGW distribution has the lowest values for the goodness-of-fit statistics (for the three data sets) among all fitted distributions. So, the TExGW distribution could be chosen as the best model. The histograms of the fitted distributions and the estimated TExGW cdf for the three data sets are displayed in Figures 5-7, respectively. The plots support the results obtained in Table 3. Figure 8 displays the QQ plots of the TExGW model for the three data sets. It is evident from these plots that the TExGW model provides good fit to the three data sets.

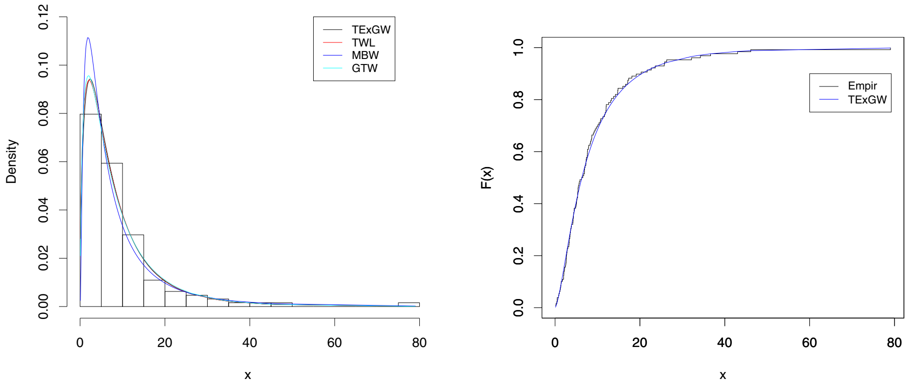

Fig. 6. The fitted pdfs (left panel) and the TExGW fitted cdf (right panel) for data set II

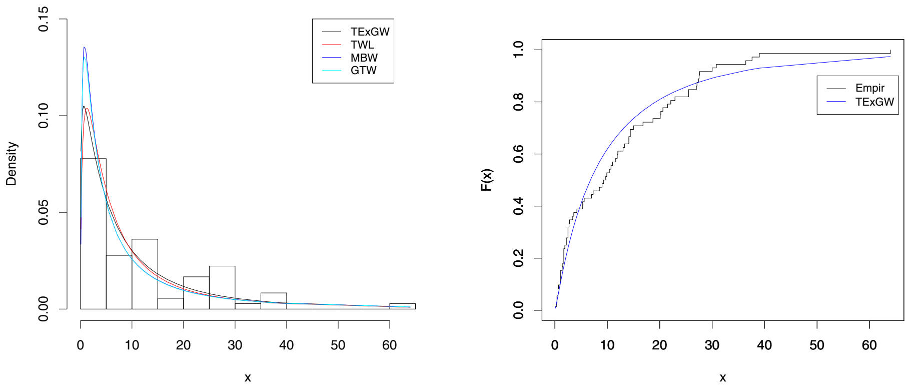

Fig. 7. The fitted pdfs (left panel) and the TExGW fitted cdf (right panel) for data set III

Table 4. MLEs and their standard errors for the data sets I, II and III.

Model	Estimates (Standard errors)				
			Data set I		
TExGW(β, a, b, λ)	0.792	0.3517	2.2103	-0.8306	
	(0.175)	(0.203)	(1.092)	(0.35)	
TWL($\alpha, \beta, a, b, \lambda$)	0.2331	7.4315	14.1387	1.4984	-0.8317
	(0.258)	(11.9819)	(39.457)	(0.633)	(0.247)
MBW(α, β, a, b, c)	1.8449	0.4477	20.6663	2.0835	2.0677
	(3.842)	(0.405)	(14.989)	(3.203)	(3.369)
GTW($\alpha, \beta, a, b, \lambda$)	5.532	0.3669	33.8286	0.0118	0.0533
	(6.208)	(0.062)	(22.276)	(0.425)	(0.915)
			Data set II		
TExGW(β, a, b, λ)	0.5542	0.7241	2.8196	-0.7497	
	(0.124)	(0.367)	(1.809)	(0.264)	
TWL($\alpha, \beta, a, b, \lambda$)	0.201	5.405	10.5705	1.5186	-0.006
	(0.18)	(5.401)	(21.344)	(0.297)	(0.505)
MBW(α, β, a, b, c)	10.1502	0.1632	57.4167	19.3859	2.0043
	(22.437)	(0.044)	(37.317)	(13.49)	(0.789)
GTW($\alpha, \beta, a, b, \lambda$)	0.2991	0.6542	2.7965	0.0128	0.002
	(0.151)	(0.121)	(1.117)	(7.214)	(1.769)
			Data set III		
TExGW(β, a, b, λ)	0.4839	0.6417	1.6013	-0.9679	
	(1.631)	(4.403)	(7.065)	(2.396)	
TWL($\alpha, \beta, a, b, \lambda$)	0.0436	1.7688	62.9424	1.6354	-0.2048
	(0.031)	(1.785)	(83.476)	(0.327)	(0.284)
MBW(α, β, a, b, c)	0.2733	0.4221	17.8045	0.3939	3.4153
	(0.011)	(0.07)	(12.156)	(0.268)	(2.14)
GTW($\alpha, \beta, a, b, \lambda$)	16.7623	0.2496	15.7843	$2.3754 \cdot 10^{-6}$	-0.0911
	(20.376)	(0.034)	(6.442)	($3.269 \cdot 10^{-4}$)	(10.123)

6. Conclusions

In this paper, we propose a new four-parameter model called the the transmuted exponentiated generalized Weibull (TExGW) distribution. The TExGW density function can be expressed as a linear mixture of exponentiated Weibull (ExW) densities. We derive explicit expressions for some of its mathematical and statistical quantities including the ordinary and incomplete moments, cumulants and probability weighted moments. We also obtain the density function of the order statistics and their moments. We discuss maximum likelihood estimation. The proposed distribution provides better fits than some other nested and nonnested models for three real data sets. We hope that the proposed model will attract wider applications in areas such as survival and lifetime data, meteorology, hydrology, engineering and others.

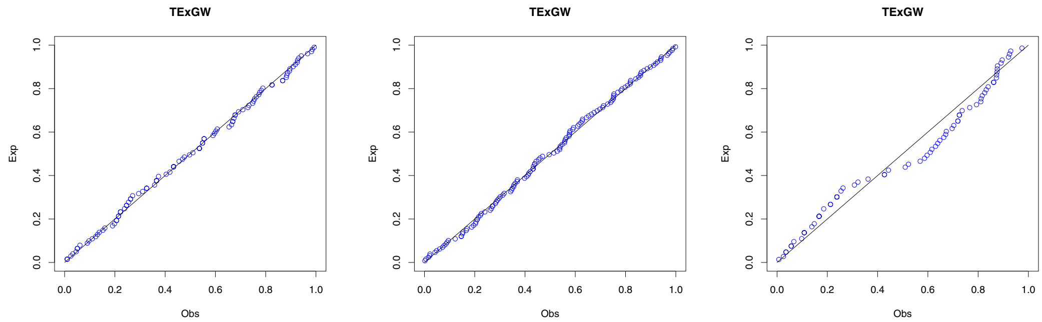

Fig. 8. QQ plots for the TExGW model for the three data sets respectively from left to right

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